



TITLE:

A CHARACTERIZATION OF $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -MIN. HYPERS IN $PG(t, q)$ ($t \geq 2$, $q \geq 5$ and $0 \leq \alpha < \beta < t$) AND ITS APPLICATIONS TO ERROR-CORRECTING CODES (Algebraic Combinatorial Theory)

AUTHOR(S):

Hamada, Noboru; Deza, Michel

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A CHARACTERIZATION OF $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -MIN-HYPERS IN $PG(t, q)$
 $(t \geq 2, q \geq 5 \text{ and } 0 \leq \alpha < \beta < t)$ AND ITS APPLICATIONS TO ERROR-CORRECTING CODES

大阪女子大学 浜田 昇 (Noboru Hamada)
 パリー大学 Michel Deza

1. Introduction

Let F be a set of f points in a finite projective geometry $PG(t, q)$ of t dimensions where $t \geq 2$, $f \geq 1$ and q is a prime power. If (a) $|F \cap H| \geq m$ for any hyperplane H in $PG(t, q)$ and (b) $|F \cap H| = m$ for some hyperplane H in $PG(t, q)$, then F is said to be an $\{f, m; t, q\}$ -min-hyper (or an $\{f, m; t, q\}$ -minihyper) where $m \geq 0$ and $|A|$ denotes the number of points in the set A . The concept of a min-hyper (or a max-hyper) has been introduced by Hamada and Tamari [17]. In the special case $t = 2$ and $m \geq 2$, an $\{f, m; 2, q\}$ -min-hyper F is called an m -blocking set if F contains no 1-flat in $PG(2, q)$.

For example, let F be a μ -flat in $PG(t, q)$ where $0 \leq \mu < t$. Then $|F| = v_{\mu+1}$ and $|F \cap H| = v_{\mu}$ or $v_{\mu+1}$ for any hyperplane H in $PG(t, q)$ according as $F \not\subset H$ or $F \subset H$ where $v_{\ell} = (q^{\ell} - 1)/(q - 1)$ for any integer $\ell \geq 0$. Hence F is a $\{v_{\mu+1}, v_{\mu}; t, q\}$ -min-hyper. Tamari [27, 29] shows that the converse holds, i.e., if F is a $\{v_{\mu+1}, v_{\mu}; t, q\}$ -min-hyper, then F is a μ -flat in $PG(t, q)$.

Let $V(n; q)$ be an n -dimensional vector space consisting of row vectors over a Galois field $GF(q)$ of order q where n is a positive integer. A k -dimensional subspace C of $V(n; q)$ is said to be an $(n, k, d; q)$ -code (or a q -ary linear code with length n , dimension k , and minimum distance d) if the minimum (Hamming) distance of the code C is equal to d where $n > k \geq 3$ and

$d \geq 1$ (cf. McWilliams and Sloane [24]). It is well known that if there exists an $(n, k, d; q)$ -code for given integers k, d and q , then

$$n \geq \sum_{\ell=0}^{k-1} \left\lceil \frac{d}{q^\ell} \right\rceil \quad (1.1)$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. In what follows, we shall confine ourselves to the case $k \geq 3$ and $1 \leq d < q^{k-1}$. In this case, d can be expressed as follows:

$$d = q^{k-1} - \sum_{\alpha=0}^{k-2} \epsilon_\alpha q^\alpha \quad (1.2)$$

using some integers k, q and ϵ_α ($\alpha = 0, 1, \dots, k-2$) and the Griesmer bound (1.1) can be expressed as follows:

$$n \geq v_k - \sum_{\alpha=0}^{k-2} \epsilon_\alpha v_{\alpha+1} \quad (1.3)$$

where $0 \leq \epsilon_\alpha \leq q-1$ for $\alpha = 0, 1, \dots, k-2$. Recently, Hamada [5, 10] showed

that in the case $k \geq 3$ and $d = q^{k-1} - \sum_{\alpha=0}^{k-2} \epsilon_\alpha q^\alpha$, there is a one-to-one

correspondence between the set of all nonequivalent $(n, k, d; q)$ -codes meeting

the Griesmer bound (1.3) and the set of all $\{ \sum_{\alpha=0}^{k-2} \epsilon_\alpha v_{\alpha+1}, \sum_{\alpha=1}^{k-2} \epsilon_\alpha v_\alpha; k-1, q \}$ -

min-hypers if we introduce some equivalence relation among $(n, k, d; q)$ -codes.

Hence in order to obtain a necessary and sufficient condition for integers

k, d and q that there exists an $(n, k, d; q)$ -code meeting the Griesmer bound

(1.3) in the case $1 \leq d < q^{k-1}$ and to characterize all $(n, k, d; q)$ -codes meet-

ing the Griesmer bound (1.3) in the case $1 \leq d < q^{k-1}$, it is sufficient to

solve the following problem.

Problem A. (1) Find a necessary and sufficient condition for integers t , q and ε_α ($\alpha = 0, 1, \dots, t-1$) that there exists a $\{\sum_{\alpha=0}^{t-1} \varepsilon_\alpha v_{\alpha+1}, \sum_{\alpha=0}^{t-1} \varepsilon_\alpha v_\alpha; t, q\}$ -min.hyper.

(2) Characterize all $\{\sum_{\alpha=0}^{t-1} \varepsilon_\alpha v_{\alpha+1}, \sum_{\alpha=0}^{t-1} \varepsilon_\alpha v_\alpha; t, q\}$ -min.hypers in the case where there exist such min.hypers.

Since all $(n, k, d; q)$ -codes meeting the Griesmer bound (1.3) have been characterized by Hellese [20] and Tilborg [30] in the special case $q = 2$, $k \geq 3$ and $1 \leq d < 2^{k-1}$, we shall confine ourself to the case $q \geq 3$, $k \geq 3$ and $1 \leq d < q^{k-1}$ in what follows.

In the case $\sum_{\alpha=0}^{k-2} \varepsilon_\alpha = 1$ (i.e., $\varepsilon_\alpha = 1$ for some integer α), it is shown by Tamari [27, 29] that F is a $\{v_{\alpha+1}, v_\alpha; k-1, q\}$ -min.hyper if and only if F is an α -flat in $PG(k-1, q)$. In the case $\sum_{\alpha=0}^{k-2} \varepsilon_\alpha = 2$, it is shown by Hamada [5, 6, 7] that F is a $\{v_{\alpha+1} + v_{\beta+1}, v_\alpha + v_\beta; k-1, q\}$ -min.hyper if and only if F is the union of an α -flat and a β -flat in $PG(k-1, q)$ which are mutually disjoint where $0 \leq \alpha \leq \beta < k-1$. In the case $\sum_{\alpha=0}^{k-2} \varepsilon_\alpha = 3$, it is shown by Hamada [5, 6, 7, 8, 9] and Hamada and Deza [14] that F is a $\{v_{\alpha+1} + v_{\beta+1} + v_{\gamma+1}, v_\alpha + v_\beta + v_\gamma; k-1, q\}$ -min.hyper if and only if F is the union of an α -flat, a β -flat and a γ -flat in $PG(k-1, q)$ which are mutually disjoint where $q \geq 5$ and either $0 \leq \alpha \leq \beta < \gamma < k-1$ or $0 \leq \alpha < \beta \leq \gamma < k-1$.

In the case $k \geq 3$, $q \geq 3$ and $\varepsilon_\alpha = 0$ or 1 for $\alpha = 0, 1, \dots, k-2$, it is shown by Hamada [5] that F is a $\{\sum_{\alpha=0}^{k-2} \varepsilon_\alpha v_{\alpha+1}, \sum_{\alpha=0}^{k-2} \varepsilon_\alpha v_\alpha; k-1, q\}$ -min.hyper if and only if F is the union of ε_0 0-flats, ε_1 1-flats, \dots , ε_{k-2} $(k-2)$ -flats in $PG(k-1, q)$ which are mutually disjoint. Hence in the case $k \geq 5$, $q \geq 3$ and

$0 \leq \alpha < \beta < \gamma < \delta < k-1$, F is a $\{v_{\alpha+1} + v_{\beta+1} + v_{\gamma+1} + v_{\delta+1}, v_{\alpha} + v_{\beta} + v_{\gamma} + v_{\delta}; k-1, q\}$ -min-hyper if and only if F is the union of an α -flat, a β -flat, a γ -flat and a δ -flat in $PG(k-1, q)$ which are mutually disjoint. Recently, it has been shown by Hamada [8] and Hamada and Deza [12] that (1) in the case $k = 3$, $q \geq 5$, $\alpha = \beta = 0$ and $\gamma = \delta = 1$, there is no $\{2v_1 + 2v_2, 2v_0 + 2v_1; 2, q\}$ -min-hyper and (2) in the case $k \geq 4$, $q \geq 5$, $\alpha = \beta = 0$ and $\gamma = \delta = 1$, F is a $\{2v_1 + 2v_2, 2v_0 + 2v_1; k-1, q\}$ -min-hyper if and only if F is the union of two 0-flats and two 1-flats in $PG(k-1, q)$ which are mutually disjoint. The purpose of this paper is to extend the above results, i.e., to prove the following theorem (cf. Reference [13] in detail).

Theorem 1.1. Let t and q be any integer ≥ 2 and any prime power ≥ 5 , respectively, and let α and β be any integers such that $0 \leq \alpha < \beta < t$.

- (1) In the case $t > 2\beta$, F is a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper if and only if F is the union of two α -flats and two β -flats in $PG(t, q)$ which are mutually disjoint.
- (2) In the case $t \leq 2\beta$, there is no $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper.

From Theorem 1.1 and Theorem 5.2 in Hamada [10], we have the following

Corollary 1.1. Let k and q be any integer ≥ 3 and any prime power ≥ 5 , respectively. Let $d = q^{k-1} - 2q^{\alpha} - 2q^{\beta}$ and $n = v_k - 2v_{\alpha+1} - 2v_{\beta+1}$ where $0 \leq \alpha < \beta < k-1$.

- (1) In the case $k > 2\beta+1$, C is an $(n, k, d; q)$ -code meeting the Griesmer bound if and only if C is an $(n, k, d; q)$ -code constructed by using two α -flats and two β -flats in $PG(k-1, q)$ which are mutually disjoint.
- (2) In the case $k \leq 2\beta+1$, there is no $(n, k, d; q)$ -code meeting (1.1).

2. Propositions for the proof of Theorem 1.1

Let $\mathcal{F}_U(\varepsilon, \mu_1, \mu_2; t, q)$ denote a family of all unions of ε points, a μ_1 -flat and a μ_2 -flat in $PG(t, q)$ which are mutually disjoint where $0 \leq \varepsilon \leq q-1$ and $1 \leq \mu_1 \leq \mu_2 < t$. Let $\mathcal{F}(v_1, v_2, \dots, v_h; t, q)$ denote a family of all unions of a v_1 -flat, a v_2 -flat, \dots , a v_h -flat in $PG(t, q)$ which are mutually disjoint where $h \geq 2$ and $0 \leq v_1 \leq v_2 \leq \dots \leq v_h < t$.

In order to prove Theorem 1.1, we shall prepare the following propositions.

Proposition 2.1. (Hamada [5,10])

Let t and q be any integer ≥ 3 and any prime power ≥ 3 , respectively, and let α and β be any integers such that $0 \leq \alpha < \beta < t/2$. If $F \in \mathcal{F}(\alpha, \alpha, \beta, \beta; t, q)$, then F is a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper.

Proposition 2.2. (Hamada [5,10])

Let t and q be any integer ≥ 2 and any prime power ≥ 3 , respectively. If there exists a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper F for some integers α and β such that $0 \leq \alpha < \beta < t$, there exists at least one $(t-2)$ -flat G in $PG(t, q)$ such that $|F \cap G| = 2v_{\alpha-1} + 2v_{\beta-1}$ where $v_{-1} = 0$ and $v_{\ell} = (q^{\ell}-1)/(q-1)$ for any integer $\ell \geq 0$. Let H_i ($i = 1, 2, \dots, q+1$) be $q+1$ hyperplanes in $PG(t, q)$ which contain G .

(1) In the case $\alpha = 0$, $F \cap H_i$ is a $\{\delta_i + 2v_{\beta}, 2v_{\beta-1}; t, q\}$ -min-hyper in H_i for $i = 1, 2, \dots, q+1$ where δ_i 's are some nonnegative integers such that $\sum_{i=1}^{q+1} \delta_i = 2$.

(2) In the case $\alpha \geq 1$, $F \cap H_i$ is a $\{2v_{\alpha} + 2v_{\beta}, 2v_{\alpha-1} + 2v_{\beta-1}; t, q\}$ -min-hyper in H_i for $i = 1, 2, \dots, q+1$.

Proposition 2.3. (Hamada [5,10])

Let t and q be any integer ≥ 4 and any prime power ≥ 3 , respectively.

(1) Let ε , β and δ_i ($i = 1, 2, \dots, q+1$) be any nonnegative integers such that $0 \leq \varepsilon \leq q-1$, $2 \leq \beta \leq t/2$ and $\sum_{i=1}^{q+1} \delta_i = \varepsilon$. If F is a $\{\varepsilon v_1 + 2v_{\beta+1}, \varepsilon v_0 + 2v_\beta; t, q\}$ -min-hyper such that (a) $|F \cap G| = 2v_{\beta-1}$ for some $(t-2)$ -flat G in $PG(t, q)$ and (b) $F \cap H_i \in \mathcal{F}_U(\delta_i, \beta-1, \beta-1; t, q)$ for any hyperplane H_i ($1 \leq i \leq q+1$) which contain G , then $F \in \mathcal{F}_U(\varepsilon, \beta, \beta; t, q)$.

(2) Let α and β be any integers such that $2 \leq \alpha < \beta \leq t/2$. If F is a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_\alpha + 2v_\beta; t, q\}$ -min-hyper such that (a) $|F \cap G| = 2v_{\alpha-1} + 2v_{\beta-1}$ for some $(t-2)$ -flat G in $PG(t, q)$ and (b) $F \cap H_i \in \mathcal{F}(\alpha-1, \alpha-1, \beta-1, \beta-1; t, q)$ for any hyperplane H_i ($1 \leq i \leq q+1$) which contain G , then $F \in \mathcal{F}(\alpha, \alpha, \beta, \beta; t, q)$.

Proposition 2.4. (Hamada and Deza [13])

Let t and q be any integer ≥ 4 and any prime power ≥ 5 , respectively, and let β be any integer such that $2 \leq \beta \leq t/2$. If F is a $\{2v_2 + 2v_{\beta+1}, 2v_1 + 2v_\beta; t, q\}$ -min-hyper such that (a) $|F \cap G| = 2v_{\beta-1}$ for some $(t-2)$ -flat G in $PG(t, q)$ and (b) $F \cap H_i \in \mathcal{F}(0, 0, \beta-1, \beta-1; t, q)$ for any hyperplane H_i ($1 \leq i \leq q+1$) which contain G , then $F \in \mathcal{F}(1, 1, \beta, \beta; t, q)$.

Proposition 2.5. (Hamada [6, 7, 10])

Let t and q be any integer ≥ 2 and any prime power ≥ 3 , respectively, and let β be an integer such that $0 \leq \beta < t$.

(1) In the case $t > 2\beta$, F is a $\{2v_{\beta+1}, 2v_\beta; t, q\}$ -min-hyper if and only if $F \in \mathcal{F}(\beta, \beta; t, q)$.

(2) In the case $t \leq 2\beta$, there is no $\{2v_{\beta+1}, 2v_\beta; t, q\}$ -min-hyper.

Proposition 2.6. (Hamada [6, 7] and Hamada and Deza [14])

Let t and q be any integer ≥ 2 and any prime power ≥ 5 , respectively, and let α and β be integers such that $0 \leq \alpha < \beta < t$.

- (1) In the case $t > 2\beta$, F is a $\{v_{\alpha+1} + 2v_{\beta+1}, v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper if and only if $F \in \mathcal{F}(\alpha, \beta, \beta; t, q)$.
- (2) In the case $t \leq 2\beta$, there is no $\{v_{\alpha+1} + 2v_{\beta+1}, v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper.

Proposition 2.7. (Hamada [8] and Hamada and Deza [12])

- (1) In the case $t = 2$ and $q \geq 5$, there is no $\{2v_1 + 2v_2, 2v_0 + 2v_1; t, q\}$ -min-hyper.
- (2) In the case $t \geq 3$ and $q \geq 5$, F is a $\{2v_1 + 2v_2, 2v_0 + 2v_1; t, q\}$ -min-hyper if and only if $F \in \mathcal{F}(0, 0, 1, 1; t, q)$.

Proposition 2.8. (Hamada and Tamari [19])

Let t and q be any integer ≥ 2 and any prime power ≥ 3 , respectively, and let α, β, γ and δ be any integers such that $0 \leq \alpha \leq \beta < \gamma \leq \delta < t$. Then $\mathcal{F}(\alpha, \beta, \gamma, \delta; t, q) \neq \emptyset$ if and only if $\gamma + \delta \leq t-1$.

3. The proof of Theorem 1.1

It follows from Proposition 2.1 that if $F \in \mathcal{F}(\alpha, \alpha, \beta, \beta; t, q)$, then F is a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper where $0 \leq \alpha < \beta < t/2$.

Suppose there exists a $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_{\alpha} + 2v_{\beta}; t, q\}$ -min-hyper F for some integer α and β such that $0 \leq \alpha < \beta < t$. Then it follows from Proposition 2.2 that there exists at least one $(t-2)$ -flat G in $PG(t, q)$ such that $|F \cap G| = 2v_{\alpha-1} + 2v_{\beta-1}$. Let H_i ($i = 1, 2, \dots, q+1$) be $q+1$ hyperplanes in $PG(t, q)$ which contain G . Then it follows from Proposition 2.2 that (1) in the case $\alpha = 0$, $F \cap H_i$ is a $\{\delta_i + 2v_{\beta}, 2v_{\beta-1}; t, q\}$ -min-hyper in H_i for $i = 1, 2, \dots, q+1$ and (2) in the case $\alpha \geq 1$, $F \cap H_i$ is a $\{2v_{\alpha} + 2v_{\beta}, 2v_{\alpha-1} + 2v_{\beta-1}; t, q\}$ -min-hyper in H_i for $i = 1, 2, \dots, q+1$ where δ_i 's are some nonnegative integers such that $\sum_{i=1}^{q+1} \delta_i = 2$. We shall prove Theorem 1.1 by induction on α and β .

Case I : $\alpha = 0$ and $\beta = 1$. It follows from Proposition 2.7 that

Theorem 1.1 holds.

Case II : $\alpha = 0$ and $\beta \geq 2$ (i.e., $\beta = \theta + 1$ and $\theta \geq 1$). Suppose Theorem 1.1

holds in the case $\alpha = 0$ and $\beta = \theta$, i.e., suppose that (1) in the case $t > 2\theta$, F is a $\{2v_1 + 2v_{\theta+1}, 2v_0 + 2v_\theta; t, q\}$ -min-hyper if and only if $F \in \mathcal{F}(0, 0, \theta, \theta; t, q)$ and (2) in the case $t \leq 2\theta$, there is no $\{2v_1 + 2v_{\theta+1}, 2v_0 + 2v_\theta; t, q\}$ -min-hyper F .

In the case $\beta = \theta + 1$, it follows from induction on β and Propositions 2.5 and 2.6 that (1) in the case $t - 1 > 2\theta$ (i.e., $t > 2\beta - 1$), $F \cap H_i$ is a $\{\delta_i v_1 + 2v_{\theta+1}, \delta_i v_0 + 2v_\theta; t, q\}$ -min-hyper in the $(t-1)$ -flat H_i if and only if $F \cap H_i$ is the union of δ_i 0-flats (i.e., δ_i points) and two θ -flats in H_i which are mutually disjoint (i.e., $F \cap H_i \in \mathcal{F}_U(\delta_i, \beta-1, \beta-1; t, q)$) and (2) in the case $t - 1 \leq 2\theta$ (i.e., $t \leq 2\beta - 1$), there is no $\{\delta_i + 2v_{\theta+1}, 2v_\theta; t, q\}$ -min-hyper in H_i . Hence it follows from Propositions 2.2 and 2.3 that (1) in the case $t > 2\beta - 1$, $F \in \mathcal{F}(0, 0, \beta, \beta; t, q)$ and (2) in the case $t \leq 2\beta - 1$, there is no $\{2v_1 + 2v_{\beta+1}, 2v_0 + 2v_\beta; t, q\}$ -min-hyper F . Since it follows from Proposition 2.8 that $\mathcal{F}(0, 0, \beta, \beta; t, q) = \emptyset$ in the case $t = 2\beta$, there is no $\{2v_1 + 2v_{\beta+1}, 2v_0 + 2v_\beta; t, q\}$ -min-hyper F in the case $t = 2\beta$. Hence Theorem 1.1 holds in Case II.

Case III : $\alpha = 1$ and $\beta \geq 2$. It follows from Cases I and II that (1) in

the case $t - 1 > 2(\beta - 1)$ (i.e., $t > 2\beta - 1$), $F \cap H_i$ is a $\{2v_1 + 2v_\beta, 2v_0 + 2v_{\beta-1}; t, q\}$ -min-hyper in the $(t-1)$ -flat H_i if and only if $F \cap H_i \in \mathcal{F}(0, 0, \beta-1, \beta-1; t, q)$ and (2) in the case $t - 1 \leq 2(\beta - 1)$ (i.e., $t \leq 2\beta - 1$), there is no $\{2v_1 + 2v_\beta, 2v_0 + 2v_{\beta-1}; t, q\}$ -min-hyper in H_i . Hence it follows from Proposition 2.4 that (1) in the case $t > 2\beta - 1$, $F \in \mathcal{F}(1, 1, \beta, \beta; t, q)$ and (2) in the case $t \leq 2\beta - 1$, there is no $\{2v_2 + 2v_{\beta+1}, 2v_1 + 2v_\beta; t, q\}$ -min-hyper F . Since it follows from Proposition 2.8 that $\mathcal{F}(1, 1, \beta, \beta; t, q) = \emptyset$ in the case $t = 2\beta$, there is no $\{2v_2 + 2v_{\beta+1}, 2v_1 + 2v_\beta; t, q\}$ -min-hyper in the case $t = 2\beta$.

Hence Theorem 1.1 holds in Case III.

Case IV : $2 \leq \alpha < \beta < t$. It follows from Propositions 2.2, 2.3 and induction on α and β that Theorem 1.1 holds. This completes the proof.

Remark 3.1. In the case $t = 2$, $\alpha = 0$, $\beta = 1$ and $q = 3$ or 4 , it is shown by Hamada [8,11] that there exists a $\{2v_1 + 2v_2, 2v_0 + 2v_1; 2, q\}$ -min-hyper F in $PG(2, q)$ such that $F \notin \mathcal{F}(0, 0, 1, 1; 2, q)$. Hence Theorem 1.1 does not hold in the case $q = 3$ or 4 .

Remark 3.2. In the case $t \geq 3$ and $q \geq 5$, we can characterize all $\{2v_{\alpha+1} + v_{\beta+1} + v_{\gamma+1}, 2v_{\alpha} + v_{\beta} + v_{\gamma}; t, q\}$ -min-hypers for any distinct integers α , β and γ in $\{0, 1, \dots, t-1\}$ using a method similar to the proof of Theorem 1.1.

Remark 3.3. In order to solve Problem A, completely, for the case $q \geq 5$ and $\sum_{\alpha=0}^{t-1} \varepsilon_{\alpha} = 3$ or 4 , it is necessary to solve the following open problem.

Problem B. Let t and q be any integer ≥ 2 and any prime power ≥ 5 , respectively.

- (1) Characterize all $\{3v_{\alpha+1}, 3v_{\alpha}; t, q\}$ -min-hypers and all $\{4v_{\alpha+1}, 4v_{\alpha}; t, q\}$ -min-hypers for any integer α in $\{1, 2, \dots, t-1\}$.
- (2) Characterize all $\{3v_{\alpha+1} + v_{\beta+1}, 3v_{\alpha} + v_{\beta}; t, q\}$ -min-hypers for any distinct integers α and β in $\{0, 1, \dots, t-1\}$.

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